XXXII Asian Pacific Mathematics Olympiad

March, 2020

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo-official.org`.
Please do not disclose nor discuss the problems online until that date. The use of calculators is not allowed.

Problem 1. Let Γ be the circumcircle of ΔABC. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Problem 2. Show that $r = 2$ is the largest real number r which satisfies the following condition:
If a sequence $a_1, a_2, \ldots$ of positive integers fulfills the inequalities

$$a_n \leq a_{n+2} \leq \sqrt{a_n^2 + r a_{n+1}}$$

for every positive integer n, then there exists a positive integer M such that $a_{n+2} = a_n$ for every $n \geq M$.

Problem 3. Determine all positive integers k for which there exist a positive integer m and a set S of positive integers such that any integer $n > m$ can be written as a sum of distinct elements of S in exactly k ways.

Problem 4. Let $\mathbb{Z}$ denote the set of all integers. Find all polynomials $P(x)$ with integer coefficients that satisfy the following property:
For any infinite sequence $a_1, a_2, \ldots$ of integers in which each integer in $\mathbb{Z}$ appears exactly once, there exist indices $i < j$ and an integer k such that $a_i + a_{i+1} + \cdots + a_j = P(k)$.

Problem 5. Let $n \geq 3$ be a fixed integer. The number 1 is written n times on a blackboard. Below the blackboard, there are two buckets that are initially empty. A move consists of erasing two of the numbers a and b, replacing them with the numbers 1 and $a + b$, then adding one stone to the first bucket and $\gcd(a, b)$ stones to the second bucket. After some finite number of moves, there are s stones in the first bucket and t stones in the second bucket, where s and t are positive integers. Find all possible values of the ratio $\dfrac{t}{s}$.