Solutions

Note: The points to be awarded for each part of the solution are indicated on the right side.

Problem 1.

$$1 = \frac{1 \times 2}{2}$$

$$1 + \frac{1}{3} = \frac{2 \times 2}{3}$$

$$1 + \frac{1}{3} + \frac{1}{6} + \dots + \frac{1}{t_n} = \frac{n \times 2}{n + 1}$$

which is easily shown by induction.

(up to 3 points)

Now S is the sum of the reciprocals of these numbers where the last, 1993006 = $\frac{1996 \times 1997}{2} = t_{1996}$. Thus we have

$$S = \frac{1}{2}\left(\frac{2}{1} + \frac{3}{2} + \dots + \frac{1997}{1996}\right)$$

$$= \frac{1}{2}\left(1996 + \left(1 + \frac{1}{2} + \dots + \frac{1}{1996}\right)\right)$$

(up to 3 points)

$$> \frac{1}{2}(1996 + 6)$$

(1 point)

$$= 1001$$

Problem 2. Note that $2^n + 2 = 2(2^{n-1} + 1)$ so that n is of the form 2r with r odd. We will consider two cases.

i) n = 2p with p prime. $2p \mid 2^{2p} + 2$, implies that $p \mid 2^{2p-1} + 1$ and hence , hence $p \mid 2^{4p-2} - 1$. On the other hand Fermat's little theorem guarantees that $p \mid 2^{p-1} - 1$. Let d = g.c.d. (p-1, 4p - 2). It follows that $p \mid 2^d - 1$. But $d \mid p - 1$ and $d \mid 4p - 2 = 4(p - 1) + 2$. Hence $d \mid 2$ and since p - 1, 4p - 2 are even d = 2. Then p = 3 and n = 6 < 100.

(up to 2 points)

ii) $n = 2pq$ where p, q are odd primes. $p < q$ and $pq < \frac{1997}{2}$. Now $n \mid 2^n + 2$ implies that $p \mid 2^{n-1} + 1$ and therefore that $p \mid 2^{2n-2} - 1 = 2^{4pq-2} - 1$. Once again by Fermat's theorem we have $p \mid 2^{n-1} - 1$ which implies that $p - 1 \mid 4pq - 2$. The same holds true for q so that

$p \mid 2^{p-1} - 1$

$$q - 1 \mid 4pq - 2 \qquad (1)$$

Both $p - 1$ and $q - 1$ are thus multiples of 2 but not of 4 so that $p \equiv q \equiv 3 \pmod 4$.

(2 points)

Taking $p = 3$, we have $4pq - 2 = 12q - 2$. Now from (1) we have

$$12 = \frac{12q - 12}{q - 1} < \frac{12q - 2}{q - 1} = \frac{12(q-1) + 10}{q - 1} = 12 + \frac{10}{q - 1} \leq 1$$

if $q \geq 11$, and clearly $\frac{12q - 2}{q - 1} = 13$ if $q = 11$. But this gives $n = 2(3)(11) = 66 < 100$. Furthermore $(p, q) = (3, 7)$ does not satisfy (1).

Taking $p = 7$ we observe that $4pq - 2 = 28q - 2$, and from (1) we have

$$28 < \frac{28q - 2}{q - 1} = \frac{28(q-1) + 26}{q - 1} = 28 + \frac{26}{q - 1} \leq 2$$

if $q \geq 27$ and clearly $\frac{28q - 2}{q - 1} = 29$ if $q = 27$. But 27 is not prime and the cases $(p, q) = (7, 11)$, $(7, 19)$ and $(7, 23)$ do not satisfy (1).

Taking $p = 11$, then $4pq - 2 = 44q - 2$, and

$$44 < \frac{44q - 2}{q - 1} \text{ and } \frac{44q - 2}{q - 1} \leq 45 \text{ if } q \geq 43.$$

Now clearly $\frac{44q - 2}{q - 1} = 45$ when $q = 43$. In this case we have $n = 2pq = 2(11)(43) = 946$. Furthermore, $\frac{2^{946} + 2}{946}$ is indeed an integer. The cases $(p, q) = (11, 19)$, $(11, 23)$ and $(11, 31)$ do not satisfy (1).

(2 points)

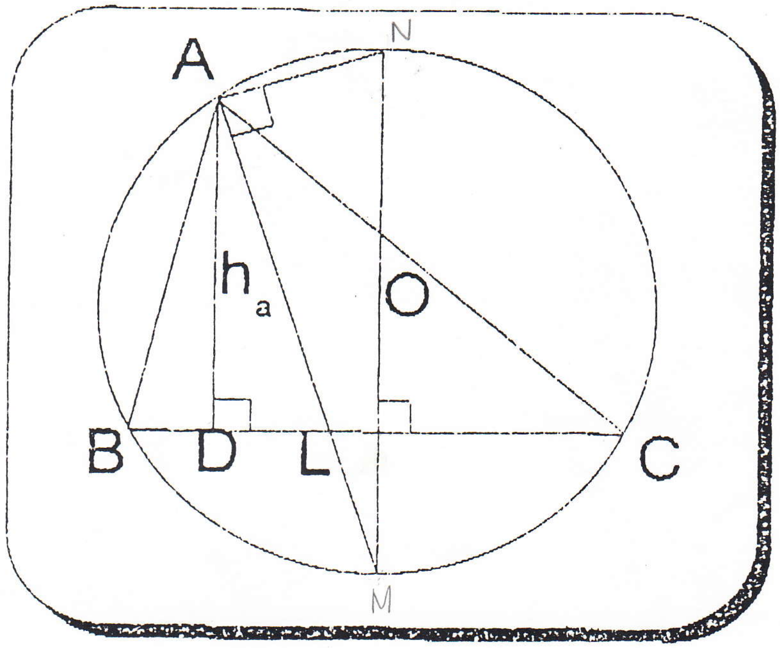
[Additionally for completeness, if $p = 19$ then $4pq - 2 = 76q - 2$ and $76 < \frac{76q - 2}{q - 1} \leq 77$ if $q \geq 75$. Now 75 is not prime and for the cases $(p, q) = (19, 23)$, $(19, 31)$, $(19, 43)$ and $(19, 47)$, $q - 1$ is not a divisor of $74 = 2 \times 37$.

Similarly, if p = 23 then 4pq - 2 = 92q - 2 and $92 < \frac{92q-2}{q-1} \leq 93$ if q ≥ 91 and $\frac{92q-2}{q-1} = 93$ if q = 91. But 91 is not prime and of the cases (p, q) = (23, 31), (23, 43), when q = 31 all of the conditions are satisfied. But, n= 2pq = 1426 is not a solution because $\frac{2^{1426}+2}{1426}$ is not an integer.

No other pairs of p, q yield numbers within the required range.]

(1 point)

Problem 3.



$\angle ALD = \frac{1}{2}(\widehat{MC} + \widehat{AB})$

$= \frac{1}{2}(\widehat{BM} + \widehat{AB})$

$= \angle ANM$

It is known (see Geometry Revisited) or easily derivable that

$$m_a^2 = (AL)^2 = bc\left(1 - \left(\frac{a}{b+c}\right)^2\right).$$

(1 point)

Using Stewart's Theorem:
Let AX be a cevian of length p, dividing BC into segments BX = m and XC = n. Then
$a(p^2 + mn) = b^2m + c^2n.$
Here, we use $m = kc$, $n = kb$, $k = \frac{a}{b+c}$

From Δ ADL ~ Δ MAN we have

$$\frac{AD}{AL} = \frac{AM}{MN} \Rightarrow AD \cdot MN = AL \cdot AM$$

$$h_a \cdot 2R = AL \cdot AM = m_a \cdot M_a$$

$$h_a = \frac{2(ABC)}{a}$$

$$\frac{2(ABC)}{a} \cdot 2R = m_a M_a \qquad (ABC) = \frac{abc}{4R}$$

$$\frac{\frac{abc}{4R} \cdot 4R}{a} = m_a M_a$$

$$bc = m_a M_a.$$

So that

$$l_a = \frac{m_a^2}{m_a M_a} = 1 - \left(\frac{a}{b+c}\right)^2$$

with similar expressions for l_b and l_c.

(2 points)

Given that $\sin A = \frac{a}{2R}$, etc. the expression we are working with becomes

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 A} + \frac{l_c}{\sin^2 C} = \frac{4R^2}{a^2}\left(1 - \left(\frac{a}{b+c}\right)^2\right) + \frac{4R^2}{b^2}\left(1 - \left(\frac{b}{a+c}\right)^2\right) + \frac{4R^2}{c^2}\left(1 - \left(\frac{c}{a+b}\right)^2\right)$$

$$= 4R^2\left[\left(\frac{1}{a^2} - \frac{1}{(b+c)^2}\right) + \left(\frac{1}{b^2} - \frac{1}{(a+c)^2}\right) + \left(\frac{1}{c^2} - \frac{1}{(a+b)^2}\right)\right]$$

$$\geq 4R^2\left[\frac{1}{a^2} - \frac{1}{4bc} + \frac{1}{b^2} - \frac{1}{4ac} + \frac{1}{c^2} - \frac{1}{4ab}\right]$$

$$= 2R^2\left[\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \left(\frac{1}{a^2} + \frac{1}{c^2}\right) + \left(\frac{1}{b^2} + \frac{1}{c^2}\right) - \frac{1}{2ab} - \frac{1}{2ac} - \frac{1}{2bc}\right]$$

$$\geq 2R^2\left[\frac{2}{ab} + \frac{2}{ac} + \frac{2}{bc} - \frac{1}{2ab} - \frac{1}{2ac} - \frac{1}{2bc}\right]$$

$$= 2R^2\left[\frac{3}{2ab} + \frac{3}{2ac} + \frac{3}{2bc}\right]$$

$$= 3R^2\left[\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right] = 3R^2\left[\frac{a+b+c}{abc}\right]$$

But $abc = 4R(ABC)$ so that this last expression becomes

$$\frac{3R(a+b+c)}{4(ABC)} = \frac{3R \cdot 2s}{4sr} = 3 \cdot \frac{R}{2r} \geq 3$$

(3 points)

since $R \geq 2r$. All of the inequalities are equalities iff $a = b = c$.

(1 point)

Problem 4.

(a) Consider the sequence of triangles on the plane $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5$, ...
It is easy to see that any pair of them are similar.
Let's prove that triangles $A_1A_3A_5$ and $A_3A_5A_7$ are similar.
Triangles $A_2A_3A_4$ and $A_4A_5A_6$ are similar and their altitudes are A_4A_5 and A_6A_7, then

$$\frac{A_2A_3}{A_4A_5} = \frac{A_4A_5}{A_6A_7} .$$

Triangles $A_3A_4A_5$ and $A_5A_6A_7$ are similar, then

$$\frac{A_4A_5}{A_6A_7} = \frac{A_3A_5}{A_5A_7}$$

Now we can conclude that

$$\frac{A_1A_3}{A_3A_5} = \frac{A_3A_5}{A_4A_5}$$

and triangles $A_1A_3A_5$ and $A_3A_5A_7$ are similar.

Hence, if P is the point where line A_1A_5 meets A_3A_7, $\angle A_3A_5A_7 = \angle PA_5A_7 = \angle A_1A_3A_5$ and $\angle A_7A_5A_1 = \angle A_7A_5P = \angle A_5A_3A_1$, so triangle A_7A_5P has a right angle at P and lines A_1A_5 and A_3A_7 are perpendicular. In the same way lines A_3A_7 and A_5A_9 are perpendicular and lines A_5A_9 and A_7A_{11} are perpendicular, hence A_1, A_5, A_9 are collinear and A_7, A_{11}, A_3 are collinear. It follows that triangle $A_1A_3A_5$ and $A_9A_{10}A_{11}$ are homothetic and the center of homothety is P. Moreover, all triangles from the family $A_1A_3A_5$, $A_9A_{10}A_{11}$, $A_{17}A_{18}A_{19}$, ...are homothetic. Of course the point P is an interior point to any of these triangles and there is no other point distinct from P that is interior to any of these triangles. So this is the point we are looking for.

(up to 4 points)

(b) Since $\angle A_1PA_3$ 90^0 then P lies on the circle with diameter A_1A_3. Let $A_1A_3 = 1$, $A_2A_3 = s$, $A_3A_5 = r$, and let $A_1A_2A_3$ be clockwise. Triangles $A_1A_2A_3$ and $A_3A_4A_5$ are similar, thus $A_4A_5 : r = s : 1$, and so $A_4A_5 = rs$. Besides $A_3A_4 = r\sqrt{1 + s^2}$ (Pythagoras), and area of triangle $A_1A_3A_5 = \frac{1}{2}r\sqrt{1 + s^2} \cdot \sqrt{1 + s^2} = \frac{1}{2}s \cdot 1$. Thus $r = \frac{s}{1 + s^2}$. By the arithmetic-geometric mean $\frac{s}{1 + s^2} \leq \frac{1}{2}$, thus $r \leq \frac{1}{2}$ and the set of all possible values of r consists of two real intervals $\left[-\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right]$. $\angle A_3A_1P$ takes the maximum value when $r = \frac{1}{2}$ thus the locus of P consists of two continuous arcs from the circle with diameter A_1A_3 with two extreme positions corresponding to $r = -\frac{1}{2}$ and $r = \frac{1}{2}$.

(up to 3 points)

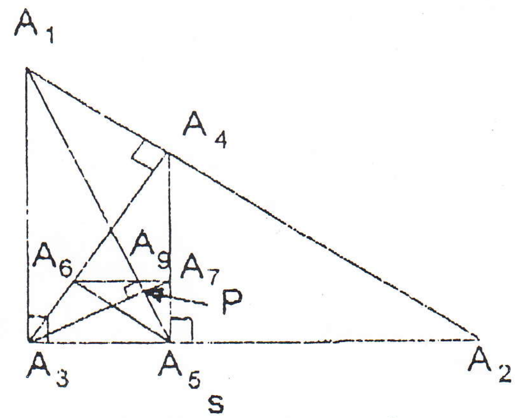


Figure showing the construction for finding point P.

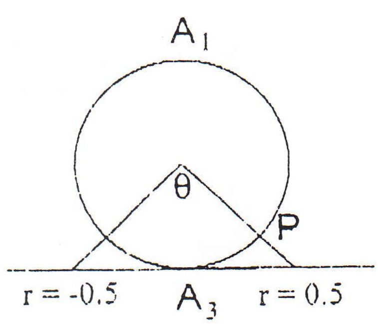


Figure showing the locus of P.

Problem 5.

A redistribution can be written as $(x_1, x_2, \cdots, x_n)$ where x_1 denotes the number of objects transferred from A_i to A_{i+1}. Our objective is to minimize the function

$$F(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} |x_1|$$

After redistribution we should have at each A_i, $a_i - x_i + x_{i-1} = N$ for $i \in \{1,2,\ldots,n\}$ where x_0 means x_n. (1 point)

Solving this system of linear equations we obtain:

$x_i = x_1 - [(i-1)N - a_2 - a_3 - \ldots - a_i]$

for $i \in \{1,2,\ldots,n\}$.

Hence

$$F(x_1, x_2, \ldots, x_n) = |x_1| + |x_1 - (N - a_2)| + |x_1 - 2N - a_2 - a_3| + \ldots + |x_1 - [(n-1)N - a_2 - a_3 - \ldots - a_n]|$$

Basically the problem reduces to find the minimum of $F(x) = \sum_{i=1}^{n} |x - \alpha_i|$

where $\alpha_i = (i-1)N - \sum_{j=2}^{i} a_j$. (up to. 3 points)

First rearrange $\alpha_1, \alpha_2, \ldots, \alpha_n$ in non decreasing order. Collecting terms which are equal to one another we write the ordered sequence $\beta_1 < \beta_2 < \cdots < \beta_m$, each β_i occurs k_i times in the family $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Thus $k_1 + k_2 + \cdots + k_m = n$.

Consider the intervals $(-\infty, \beta_1], [\beta_1, \beta_2], \cdots, [\beta_{m-1}, \beta_m], [\beta_m, \infty)$ the graph of $F(x) = \sum_{i=1}^{n} |x - \alpha_i| = \sum_{i=1}^{m} k_i |x - \beta_i|$ is a continuos piece wise linear graph define in the following way:

$$F(x) = \begin{cases} k_1(\beta_1 - x) + k_2(\beta_2 - x) + \cdots + k_m(\beta_m - x) \text{ if } x \in (-\infty, \beta_1] \\ k_1(x - \beta_1) + k_2(\beta_2 - x) + \cdots + k_m(\beta_m - x) \text{ if } x \in [\beta_1, \beta_2] \\ \vdots \\ k_1(x - \beta_1) + k_2(x - \beta_2 + \cdots + k_m)(x - \beta_m) \text{ if } x \in [\beta_m, \infty) \end{cases}$$

(up to 4 points)

The slopes of each line segment on each interval are respectively:

$S_0 = -k_1 - k_2 - k_3 - \cdots - k_m$

$S_1 = k_1 - k_2 - k_3 - \cdots - k_m$

$S_2 = k_1 + k_2 - k_3 - \cdots - k_m$

$S_m = k_1 + k_2 + k_3 + \cdots + k_m$

Note that this sequence of increasing numbers goes from a negative to a positive number, hence for some $t \geq 1$ there is an

$$S_t = 0 \text{ or } S_{t-1} < 0 < S_t$$

In the first case the minimum occurs at $x = \beta_t$ or β_{t+1} and in the second case the minimum occurs at $x = \beta_t$

(Up to 7 points)

We can rephrase the computations above in terms of $\alpha_1, \alpha_2, \cdots, \alpha_n$ rather than $\beta_1, \beta_2, \cdots, \beta_m$. After rearranging the α's in non decreasing order, pick $x = \alpha_{\frac{n+1}{2}}$ if n is odd and take $x = \alpha_{\frac{n}{2}}$ or $\alpha_{\frac{n}{2}+1}$ if n is even.

If no justification is given for the choice of x , give up to 4 points.