XXXIII Asian Pacific Mathematics Olympiad

March, 2021

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo-official.org`.
Please do not disclose nor discuss the problems online until that date. The use of calculators is not allowed.

Problem 1. Prove that for each real number $r > 2$, there are exactly two or three positive real numbers x satisfying the equation $x^2 = r\lfloor x \rfloor$.
Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Problem 2. For a polynomial P and a positive integer n, define P_n as the number of positive integer pairs (a, b) such that $a < b \leq n$ and $|P(a)| - |P(b)|$ is divisible by n. Determine all polynomial P with integer coefficients such that $P_n \leq 2021$ for all positive integers n.

Problem 3. Let $ABCD$ be a cyclic convex quadrilateral and Γ be its circumcircle. Let E be the intersection of the diagonals AC and BD, let L be the center of the circle tangent to sides AB, BC, and CD, and let M be the midpoint of the arc BC of Γ not containing A and D. Prove that the excenter of triangle BCE opposite E lies on the line LM.

Problem 4. Given a 32×32 table, we put a mouse (facing up) at the bottom left cell and a piece of cheese at several other cells. The mouse then starts moving. It moves forward except that when it reaches a piece of cheese, it eats a part of it, turns right, and continues moving forward. We say that a subset of cells containing cheese is good if, during this process, the mouse tastes each piece of cheese exactly once and then falls off the table. Show that:

(a) No good subset consists of 888 cells.

(b) There exists a good subset consisting of at least 666 cells.

Problem 5. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f\left(f(a) - b\right) + bf(2a)$ is a perfect square for all integers a and b.