XXXIV Asian Pacific Mathematics Olympiad

March, 2022

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo-official.org`.
Please do not disclose nor discuss the problems online until that date. The use of calculators is not allowed.

Problem 1. Find all pairs (a, b) of positive integers such that a^3 is a multiple of b^2 and $b - 1$ is a multiple of $a - 1$.
Note: An integer n is said to be a multiple of an integer m if there is an integer k such that $n = km$.

Problem 2. Let ABC be a right triangle with $\angle B = 90°$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the second intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.

Problem 3. Find all positive integers $k < 202$ for which there exists a positive integer n such that

$$\left\{\frac{n}{202}\right\} + \left\{\frac{2n}{202}\right\} + \cdots + \left\{\frac{kn}{202}\right\} = \frac{k}{2},$$

where $\{x\}$ denote the fractional part of x.
Note: The fractional part of a real number x is defined as the real number k with $0 \leq k < 1$ such that $x - k$ is an integer.

Problem 4. Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labelled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i + 1$. Cathy wins if at any point there is a box containing only marble n. Determine all pairs of integers (n, k) such that Cathy can win this game.

Problem 5. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Determine the minimum value of $(a-b)(b-c)(c-d)(d-a)$ and determine all values of (a, b, c, d) such that the minimum value is achieved.