XXXVII Asian Pacific Mathematics Olympiad

March, 2025

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo-official.org`*. Please do not disclose nor discuss the problems online until that date. The use of calculators is not allowed.*

Problem 1. Let ABC be an acute triangle inscribed in a circle Γ. Let A_1 be the orthogonal projection of A onto BC so that AA_1 is an altitude. Let B_1 and C_1 be the orthogonal projections of A_1 onto AB and AC, respectively. Point P is such that quadrilateral AB_1PC_1 is convex and has the same area as triangle ABC. Is it possible that P strictly lies in the interior of circle Γ? Justify your answer.

Problem 2. Let α and β be positive real numbers. *Emerald* makes a trip in the coordinate plane, starting off from the origin $(0,0)$. Each minute she moves one unit up or one unit to the right, restricting herself to the region $|x-y|<2025$, in the coordinate plane. By the time she visits a point (x,y) she writes down the integer $\lfloor x\alpha + y\beta \rfloor$ on it. It turns out that Emerald wrote each non-negative integer exactly once. Find all the possible pairs (α,β) for which such a trip would be possible.

Problem 3. Let $P(x)$ be a non-constant polynomial with integer coefficients such that $P(0)\neq 0$. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of integers such that $P(i-j)$ divides $a_i - a_j$ for all distinct positive integers i, j. Prove that the sequence $a_1, a_2, a_3, \ldots$ must be constant, that is, a_n equals a constant c for all n positive integer.

Problem 4. Let $n \geq 3$ be an integer. There are n cells on a circle, and each cell is assigned either 0 or 1. There is a rooster on one of these cells, and it repeats the following operation:

- If the rooster is on a cell assigned 0, it changes the assigned number to 1 and moves to the next cell counterclockwise.
- If the rooster is on a cell assigned 1, it changes the assigned number to 0 and moves to the cell after next cell counterclockwise.

Prove that the following statement holds after sufficiently many operations:

> If the rooster is on a cell C, then the rooster would go around the circle exactly three times, stopping again at C. Moreover, every cell would be assigned the same number as it was assigned right before the rooster went around the circle 3 times.

Problem 5. Consider an infinite sequence $a_1, a_2, \ldots$ of positive integers such that

$$100!(a_m + a_{m+1} + \cdots + a_n) \quad \text{is a multiple of} \quad a_{n-m+1}a_{n+m}$$

for all positive integers m, n such that $m \leq n$.

Prove that the sequence is either bounded or linear.

Observation: A sequence of positive integers is *bounded* if there exists a constant N such that $a_n < N$ for all $n \in \mathbb{Z}_{>0}$. A sequence is *linear* if $a_n = n \cdot a_1$ for all $n \in \mathbb{Z}_{>0}$.