APMO 1989 – Problems and Solutions

Problem 1
Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \le 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!}.$$

Solution 1
Let σ_k be the *kth symmetric polynomial*, namely

$$\sigma_k = \sum_{\substack{|S|=k \\ S\subseteq\{1,2,\ldots,n\}}} \prod_{i\in S} x_i,$$

and more explicitly

$$\sigma_1 = S, \quad \sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \quad \text{and so on.}$$

Then

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n.$$

The expansion of

$$S^k = (x_1 + x_2 + \cdots + x_n)^k = \underbrace{(x_1 + x_2 + \cdots + x_n)(x_1 + x_2 + \cdots + x_n) \cdots (x_1 + x_2 + \cdots + x_n)}_{k \text{ times}}$$

has at least $k!$ occurrences of $\prod_{i\in S} x_i$ for each subset S with k indices from $\{1, 2, \ldots, n\}$. In fact, if π is a permutation of S, we can choose each $x_{\pi(i)}$ from the ith factor of $(x_1 + x_2 + \cdots + x_n)^k$. Then each term appears at least $k!$ times, and

$$S^k \ge k!\sigma_k \iff \sigma_k \le \frac{S^k}{k!}.$$

Summing the obtained inequalities for $k = 1, 2, \ldots, n$ yields the result.

Solution 2
By AM-GM,

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \le \left(\frac{(1 + x_1) + (1 + x_2) + \cdots + (1 + x_n)}{n}\right)^n = \left(1 + \frac{S}{n}\right)^n.$$

By the binomial theorem,

$$\left(1 + \frac{S}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{S}{n}\right)^k = \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)\ldots(n-k+1)}{n^k} S^k \le \sum_{k=0}^{n} \frac{S^k}{k!},$$

and the result follows.

Comment: Maclaurin's inequality states that

$$\frac{\sigma_1}{n} \ge \sqrt{\frac{\sigma_2}{\binom{n}{2}}} \ge \cdots \ge \sqrt[k]{\frac{\sigma_k}{\binom{n}{k}}} \ge \cdots \ge \sqrt[n]{\frac{\sigma_n}{\binom{n}{n}}}.$$

Then $\sigma_k \le \binom{n}{k}\frac{S^k}{n^k} = \frac{1}{k!}\frac{n(n-1)\ldots(n-k+1)}{n^k}S^k \le \frac{S^k}{k!}$.

Problem 2

Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

Solution

We can suppose without loss of generality that $a, b, c, n \geq 0$. Let (a, b, c, n) be a solution with minimum sum $a + b + c + n$. Suppose, for the sake of contradiction, that $a + b + c + n > 0$. Since 6 divides $5n^2$, n is a multiple of 6. Let $n = 6n_0$. Then the equation reduces to

$$6a^2 + 3b^2 + c^2 = 30n_0^2.$$

The number c is a multiple of 3, so let $c = 3c_0$. The equation now reduces to

$$2a^2 + b^2 + 3c_0^2 = 10n_0^2.$$

Now look at the equation modulo 8:

$$b^2 + 3c_0^2 \equiv 2(n_0^2 - a^2) \pmod 8.$$

Integers b and c_0 have the same parity. Either way, since x^2 is congruent to 0 or 1 modulo 4, $b^2 + 3c_0^2$ is a multiple of 4, so $n_0^2 - a^2 = (n_0 - a)(n_0 + a)$ is even, and therefore also a multiple of 4, since $n_0 - a$ and $n_0 + a$ have the same parity. Hence $2(n_0^2 - a^2)$ is a multiple of 8, and

$$b^2 + 3c_0^2 \equiv 0 \pmod 8.$$

If b and c_0 are both odd, $b^2 + 3c_0^2 \equiv 4 \pmod 8$, which is impossible. Then b and c_0 are both even. Let $b = 2b_0$ and $c_0 = 2c_1$, and we find

$$a^2 + 2b_0^2 + 6c_1^2 = 5n_0^2.$$

Look at the last equation modulo 8:

$$a^2 + 3n_0^2 \equiv 2(c_1^2 - b_0^2) \pmod 8.$$

A similar argument shows that a and n_0 are both even.

We have proven that a, b, c, n are all even. Then, dividing the original equation by 4 we find

$$6(6(a/2)^2 + 3(b/2)^2 + (c/2)^2) = 5(n/2)^2,$$

and we find that $(a/2, b/2, c/2, n/2)$ is a new solution with smaller sum. This is a contradiction, and the only solution is $(a, b, c, n) = (0, 0, 0, 0)$.

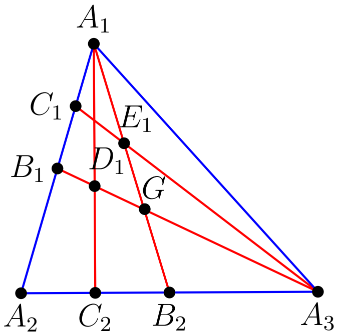
Problem 3

Let A_1, A_2, A_3 be three points in the plane, and for convenience,let $A_4 = A_1$, $A_5 = A_2$. For $n = 1, 2$, and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Answer: $\dfrac{25}{49}$.

Solution

Let G be the centroid of triangle ABC, and also the intersection point of A_1B_2, A_2B_3, and A_3B_1.



By Menelao's theorem on triangle $B_1A_2A_3$ and line $A_1D_1C_2$,

$$\frac{A_1B_1}{A_1A_2} \cdot \frac{D_1A_3}{D_1B_1} \cdot \frac{C_2A_2}{C_2A_3} = 1 \iff \frac{D_1A_3}{D_1B_1} = 2 \cdot 3 = 6 \iff \frac{D_1B_1}{A_3B_1} = \frac{1}{7}.$$

Since $A_3G = \frac{2}{3}A_3B_1$, if $A_3B_1 = 21t$ then $GA_3 = 14t$, $D_1B_1 = \frac{21t}{7} = 3t$, $A_3D_1 = 18t$, and $GD_1 = A_3D_1 - A_3G = 18t - 14t = 4t$, and

$$\frac{GD_1}{GA_3} = \frac{4}{14} = \frac{2}{7}.$$

Similar results hold for the other medians, therefore $D_1D_2D_3$ and $A_1A_2A_3$ are homothetic with center G and ratio $-\frac{2}{7}$.
By Menelao's theorem on triangle $A_1A_2B_2$ and line $C_1E_1A_3$,

$$\frac{C_1A_1}{C_1A_2} \cdot \frac{E_1B_2}{E_1A_1} \cdot \frac{A_3A_2}{A_3B_2} = 1 \iff \frac{E_1B_2}{E_1A_1} = 3 \cdot \frac{1}{2} = \frac{3}{2} \iff \frac{A_1E_1}{A_1B_2} = \frac{2}{5}.$$

If $A_1B_2 = 15u$, then $A_1G = \frac{2}{3} \cdot 15u = 10u$ and $GE_1 = A_1G - A_1E_1 = 10u - \frac{2}{5} \cdot 15u = 4u$, and

$$\frac{GE_1}{GA_1} = \frac{4}{10} = \frac{2}{5}.$$

Similar results hold for the other medians, therefore $E_1E_2E_3$ and $A_1A_2A_3$ are homothetic with center G and ratio $\frac{2}{5}$.
Then $D_1D_2D_3$ and $E_1E_2E_3$ are homothetic with center G and ratio $-\frac{2}{7} : \frac{2}{5} = -\frac{5}{7}$, and the ratio of their area is $\left(\frac{5}{7}\right)^2 = \frac{25}{49}$.

Problem 4
Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \le a < b \le n$. Show that there are at least

$$4m\frac{(m - \frac{n^2}{4})}{3n}$$

triples (a, b, c) such that (a, b), (a, c), and (b, c) belong to S.

Solution
Call a triple (a, b, c) *good* if and only if (a, b), (a, c), and (b, c) all belong to S. For i in $\{1, 2, \dots, n\}$, let d_i be the number of pairs in S that contain i, and let D_i be the set of numbers paired with i in S (so $|D_i| = d_i$). Consider a pair $(i, j) \in S$. Our goal is to estimate the number of integers k such that any permutation of $\{i, j, k\}$ is good, that is, $|D_i \cap D_j|$. Note that $i \notin D_i$ and $j \notin D_j$, so $i, j \notin D_i \cap D_j$; thus any $k \in D_i \cap D_j$ is different from both i and j, and $\{i, j, k\}$ has three elements as required. Now, since $D_i \cup D_j \subseteq \{1, 2, \dots, n\}$,

$$|D_i \cap D_j| = |D_i| + |D_j| - |D_i \cup D_j| \le d_i + d_j - n.$$

Summing all the results, and having in mind that each good triple is counted three times (one for each two of the three numbers), the number of good triples T is at least

$$T \ge \frac{1}{3} \sum_{(i,j)\in S} (d_i + d_j - n).$$

Each term d_i appears each time i is in a pair from S, that is, d_i times; there are m pairs in S, so n is subtracted m times. By the Cauchy-Schwartz inequality

$$T \ge \frac{1}{3}\left(\sum_{i=1}^{n} d_i^2 - mn\right) \ge \frac{1}{3}\left(\frac{\left(\sum_{i=1}^{n} d_i\right)^2}{n} - mn\right).$$

Finally, the sum $\sum_{i=1}^{n} d_i$ is $2m$, since d_i counts the number of pairs containing i, and each pair (i, j) is counted twice: once in d_i and once in d_j. Therefore

$$T \ge \frac{1}{3}\left(\frac{(2m)^2}{n} - mn\right) = 4m\frac{(m - \frac{n^2}{4})}{3n}.$$

Comment: This is a celebrated graph theory fact named *Goodman's bound*, after A. M. Goodman's method published in 1959. The generalized version of the problem is still studied to this day.

Problem 5
Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing,

(2) $f(x) + g(x) = 2x$ for all real x, where $g(x)$ is the composition inverse function to $f(x)$.

(*Note*: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

Answer: $f(x) = x + c$, $c \in \mathbb{R}$ constant.

Solution
Denote by f_n the nth iterate of f, that is, $f_n(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)))$.

Plug $x \to f_{n+1}(x)$ in (2): since $g(f_{n+1}(x)) = g(f(f_n(x))) = f_n(x)$,

$$f_{n+2}(x) + f_n(x) = 2f_{n+1}(x),$$

that is,

$$f_{n+2}(x) - f_{n+1}(x) = f_{n+1}(x) - f_n(x).$$

Therefore $f_n(x) - f_{n-1}(x)$ does not depend on n, and is equal to $f(x) - x$. Summing the corresponding results for smaller values of n we find

$$f_n(x) - x = n(f(x) - x).$$

Since g has the same properties as f,

$$g_n(x) - x = n(g(x) - x) = -n(f(x) - x).$$

Finally, g is also increasing, because since f is increasing $g(x) > g(y) \implies f(g(x)) > f(g(y)) \implies x > y$. An induction proves that f_n and g_n are also increasing functions.
Let $x > y$ be real numbers. Since f_n and g_n are increasing,

$$x + n(f(x) - x) > y + n(f(y) - y) \iff n[(f(x) - x) - (f(y) - y)] > y - x$$

and

$$x - n(f(x) - x) > y - n(f(y) - y) \iff n[(f(x) - x) - (f(y) - y)] < x - y.$$

Summing it up,

$$|n[(f(x) - x) - (f(y) - y)]| < x - y \quad \text{for all } n \in \mathbb{Z}_{>0}.$$

Suppose that $a = f(x) - x$ and $b = f(y) - y$ are distinct. Then, for all positive integers n,

$$|n(a - b)| < x - y,$$

which is false for a sufficiently large n. Hence $a = b$, and $f(x) - x$ is a constant c for all $x \in \mathbb{R}$, that is, $f(x) = x + c$.
It is immediate that $f(x) = x + c$ satisfies the problem, as $g(x) = x - c$.