SOLUTIONS

Note: On the left side of the page the maximum number of points that may be awarded for every part of the solution is indicated in brackets.

Question 1. Suppose that $(a_1, a_2, \ldots, a_{1995})$ is a solution of the given system of inequalities. Then

$$\sum_{n=1}^{1995} 2\sqrt{a_n - (n-1)} \geq \sum_{n=1}^{1995} a_n - \sum_{n=1}^{1994} (n-1) + 1 = \sum_{n=1}^{1995} a_n - \sum_{n=1}^{1995} \{(n-1) + 1\}$$

i.e.

$$0 \geq \sum_{n=1}^{1995} \{a_n - (n-1) + 1 - 2\sqrt{a_n - (n-1)} \;.$$

[2 points]

Next, note that

$$\left[\sqrt{a_n - (n-1)} - 1\right]^2 = a_n - (n-1) - 2\sqrt{a_n - (n-1)} + 1$$

for n = 1, 2, ..., 1995.

[1 point]

Hence,

$$0 \geq \sum_{n=1}^{1995} \left[a_n - (n-1) - 2\sqrt{a_n - (n-1)} + 1\right] = \sum_{n=1}^{1995} \left[\sqrt{a_n - (n-1)} - 1\right]^2 \geq 0 .$$

Therefore, $\sqrt{a_n - (n-1)} = 1$, for n = 1, 2, ..., 1995. It follows that $a_n = n$ for n = 1, 2, ..., 1995.

[2 points]

Conversely, if $\sqrt{a_n - (n-1)} = 1$, for n = 1, 2, ..., 1995, then

$$2\sqrt{n - (n-1)} = 2 = n + 1 - (n-1) \text{, for n = 1, 2, ..., 1994}$$

and

$$2\sqrt{1995 - 1994} = 2 = 1 + 1,$$

which shows that $a_n = n$, for n = 1, 2, ..., 1995, is indeed a solution of the given system of inequalities.

[2 points]

Question 2. The answer is 14.

[1 point]

Denote the required number by M. We observe that the sequence 2·101, 3·97, 5·89, 7·83, 11·79, 13·73, 17·71, 19·67, 23·61 = 1403, 29·59 = 1711, 31·53 = 1643, 37·47 = 1739, 41·43 = 1763 satisfies conditions i) and ii) and contains no prime number. Hence, M > 13.

[3 points]

Now we show that a sequence with 14 elements that satisfies conditions i) and ii) will contain a prime number. We proceed by contradiction. Suppose the elements are $a_1, a_2, \ldots, a_{14}$. Since none of them is a prime number, each element will contain at least two prime factors. We take any two prime factors from each a_i, and list them in ascending order $p_1 < p_2 < \ldots < p_{26} < p_{27} < p_{28}$. As the 14th prime is 43, this means $43 \le p_{14}$, $47 \le p_{15}$ and so on. Now 43·47 = 2021 > 1995. This means that p_{14} must pair up with one of the $p_1, p_2 \ldots p_{13}$ to form a certain a_i. Likewise p_{15} must pair up with one of the $p_1, p_2 \ldots p_{13}$ to form another a_i, and so on (without repetition). Hence there exist p_i, p_j, $13 < i < j$, that must pair up together to form some a_i. But then $a_i \ge p_i p_j \ge 43 \cdot 47 > 1995$, a contradiction.

[3 points]

Question 3. Let T be the intersection of PQ and RS, T lies outside C, the circle PQRS.

i) Clearly any point on C belongs to the set A.
ii) Let $r = \sqrt{TP \cdot TQ} = \sqrt{TR \cdot TS}$, and consider the circle with center T and radius r. Let V a point on this circle. Since $TV^2 = TP \cdot TQ = TR \cdot TS$, TV is tangent to the circles PQV and RSV. Therefore, PQV is tangent to RSV. That means, V is in the set A.

[4 points]

Conversely, assume V is in A, i.e. PQV is tangent to RSV. If the circles PQV and RSV are the same, then PQV = RSV = PQRS. Otherwise, let the line TV intersect PQV in V_1, and RSV in V_2. Then

$$TP \cdot TQ = TV \cdot TV_1$$
$$TR \cdot TS = TV \cdot TV_2 .$$

Due to the fact that PQR and S are on a circle, we have TP·TQ = TR·TS, thus $TV \cdot TV_1 = TV \cdot TV_2$. Moreover, since T does not lie on C, T ≠ V, which implies $TV_1 = TV_2$, i.e., $V_1 = V_2 = V$.
All this means that TV is tangent to the circles PQV and RSV, therefore V lies on the circle with center T and radius $r = \sqrt{TP \cdot TQ} = \sqrt{TR \cdot TS}$.

[3 points]

Question 4. First, we will show that MS is perpendicular to A'B'. Since SAMB, SBN'A', SA'M'B' and SB'NA are rectangles, it follows that MNM'N' is a rectangle with its sides parallel to AA' and BB'.
Moreover, the perpendicular bisectors of AA' and BB' pass through O, and they coincide with those of MN' and NM'. Therefore, O is the center of the rectangle.
Let I and H be the intersections of MS with AB and A'B'. We then have
$\angle HSA' = \angle ASI$,
$\angle ASI = \angle SAI$,
$\angle SAI = \angle A'AB = \angle A'B'B$.
In the triangle SA'B', $\angle A'B'B$ or $\angle A'B'S$ is the complementary angle of $\angle SA'B'$.
The angles HSA' and SA'B are complementary angles and the triangle SA'H is a right-angled triangle with right angle at H. Therefore, MS ⊥ A'B'.

[1 point]

Next, we will show that $AB^2 + A'B'^2 = 4R^2$ and that $MN'^2 + N'M'^2$ is constant.

Let D be the second intersection of MN' with the circle, then A'D = AB, since they subtend equal angles. This implies

$$AB^2 + A'B'^2 = A'D^2 + A'B'^2 .$$

But, we know DA' || MH, since $\angle BDA' = \angle BAA' = \angle BMH$, that means $\angle DA'B' = 90^0$ and it is inscribed in the circle, therefore D and B' are diametrically opposed, what finally implies

$$AB^2 + A'B'^2 = A'D^2 + A'B'^2 = DB'^2 = (2R)^2 = 4R^2,$$

i.e.

$$AB^2 + A'B'^2 = 4R^2.$$

[2 points]

To see that $MN'^2 + N'M'^2$ is constant consider the following equalities

$$MN'^2 = (MB + BN')^2 = MB^2 + BN'^2 + 2MB \cdot BN'$$
$$= SA^2 + SA'^2 + 2SA \cdot SA'^2$$
$$M'N'^2 = (N'A' + A'M')^2 = N'A'^2 + A'M'^2 + 2N'A' \cdot A'M'$$
$$= SB^2 + SB'^2 + 2SB \cdot SB'.$$

By Pythagoras, we have

$$AB^2 + A'B'^2 = (SA^2 + SB^2) + (SA'^2 + SB'^2)$$

This implies,

$$MN'^2 + M'N'^2 = SA^2 + SB^2 + SA'^2 + SB'^2 + 2SA\cdot SA' + 2SBSB'$$
$$= AB^2 + A'B'^2 + 4SA\cdot SA'$$
$$= 8R^2 - 4OS^2.$$

Additionally we know that

$$MN'^2 + M'N'^2 = MM'^2 = 4OM^2.$$

[2 points]

But, $4OM^2 = 8R^2 - 4OS^2$.
Therefore,

$$MN'^2 + M'N'^2 = 4OM^2$$

This last quantity is clearly a constant.

[1 point]

Finally, it is clear that the vertices of the rectangle MNM'N' lie on the circle with center O and radius OM = $\sqrt{2R^2 - OS^2}$. Therefore, the set of points consists of a circle.

[1 point]

Question 5. The minimum value of k is $k^* = 4$.

[1 point]

First, we define a function f from Z to {1, 2, 3, 4} recursively as follows: f(0) = 1. For any positive integer i, f(i) is defined to be the minimum positive integer not in $A_i := \{f(j) : i - j \in \{5, 7, 12\} \text{ and } -i < j < i\}$, and f(-i) the minimum positive integer not in $B_i := \{f(j) : j + i \in \{5, 7, 12\} \text{ and } -i < j < i\}$. Note that $|A_i| \leq 3$ and $|B_i| \leq 3$ for any i. So, f is a function from Z to {1, 2, 3, 4} such that $f(x) \neq f(y)$ whenever $|x - y| \in \{5, 7, 12\}$. This gives that $k^* \leq 4$.

[3 points]

Next, we claim that $k^* \geq 4$. Suppose it is not the case. Then there exists a function f from Z to {1, 2, 3} with the property that $f(x) \neq f(y)$ whenever $|x - y| \in \{5, 7, 12\}$. For any integer x, consider the values f(x), f(x - 5), and f(x + 7). These three values are different. Now consider f(x + 2). Since $f(x + 2) \notin \{f(x - 5), f(x + 7)\}$

$$f(x) = f(x + 2) \text{ for any integer } x.$$

Hence,

$$f(x) = f(x + 2) = f(x + 4) = \ldots = f(x + 12),$$

which is impossible. Thus $k^* > 4$.

[3 points]