XI APMO - SOLUTIONS AND MARKING SCHEMES

Problem 1.

First solution.

Let us prove that $n = S(n) + 9T(n)$ proceding by induction over the number of digits of n.

1 POINT.

If n has one digit then the result is trivial. Suppose that $n = S(n) + 9T(n)$ is true for any integer n of k digits. Now any number m of $k+1$ digits can be writen as $m = 10n + a$ where n is a number of k digits. Obviously,

$$T(m) = n + T(n) \quad \text{and} \quad (m) = S(n) + a.$$

2 POINTS (1 POINT for each of the last equalities).

Therefore

$$\begin{aligned} m - S(m) &= 10n + a - S(n) - a \\ &= 10n - S(n) \\ &= (n - S(n)) + 9n \\ &= 9T(n) + 9n \\ &= 9T(m), \end{aligned}$$

as required.

4 POINTS.

Second solution.

Let $n = \overline{a_k a_{k-1} \ldots a_1 a_0} = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10a_1 + a_0$, where a_k, a_{k-1}, $\ldots$, a_1, a_0 are digits. The stumps of n are

$$\begin{aligned} \overline{a_k a_{k-1} \ldots a_1} &= 10^{k-1} a_k + 10^{k-2} a_{k-1} + \ldots + 10a_2 + a_1 \\ \overline{a_k a_{k-1} \ldots a_2} &= 10^{k-2} a_k + 10^{k-3} a_{k-1} + \ldots + a_2 \\ &\vdots \\ \overline{a_k a_{k-1}} &= 10a_k + a_{k-1} \\ a_k &= a_k. \end{aligned}$$

1 POINT.

Since $10^{m-1} + 10^{m-2} + \ldots + 10 + 1 = \frac{10^m - 1}{9}$ then the sum of all stumps of n is

$$T(n) = \frac{10^k - 1}{9} a_k + \frac{10^{k-1} - 1}{9} a_{k-1} + \ldots + \frac{10 - 1}{9} a_1,$$

3 POINTS.

and hence

$$\begin{aligned} 9T(n) &= 10^k a_k + 10^{k-1}a_{k-1} + \ldots + 10a_1 + a_0 - (a_k + a_{k-1} + \ldots + a_1 + a_0) \\ &= n - S(n). \end{aligned}$$

Consequently $n = S(n) + 9T(n)$.

3 POINTS.

Third Solution.

Let $k(n)$ be the number of zeros at the end of the decimal representation of n or, which is the same, the largest power of 10 which divides n. The the following two observations are straightforward:

1. $S(n) = S(n-1) - (9k(n) - 1)$,
2. $k(1) + k(2) + \ldots + k(n) = T(n)$.

4 POINTS (2 POINTS for each of these equalities).

Then summing the following equalities up

$$\begin{aligned} S(1) &= S(0) - (9k(1) - 1), \\ S(2) &= S(1) - (9k(2) - 1), \\ &\vdots \\ S(n-1) &= S(n-2) - (9k(n-1) - 1), \\ S(n) &= S(n-1) - (9k(n-1) - 1) \end{aligned}$$

we get $S(n) = n - (k(1) + k(2) + \ldots + k(n)) = n - 9T(n)$.

3 POINT for concluding.

Problem 2.

First Solution.

This is equivalent to find the largest positive integer solution of the equation

$$\left\lfloor \frac{N}{3} \right\rfloor = \left\lfloor \frac{N}{5} \right\rfloor + \left\lfloor \frac{N}{7} \right\rfloor - \left\lfloor \frac{N}{35} \right\rfloor,$$

or

$$\left\lfloor \frac{N}{3} \right\rfloor + \left\lfloor \frac{N}{35} \right\rfloor = \left\lfloor \frac{N}{5} \right\rfloor + \left\lfloor \frac{N}{7} \right\rfloor. \tag{1}$$

1 POINT for equality (1).

For N to be a solution of (1) it is necessary that

$$\frac{N-2}{3} + \frac{N-34}{35} \le \frac{N}{5} + \frac{N}{7},$$

which simplifies to $N \le 86$.

1 POINTS for finding that $N \le 86$.

However, if $N \ge 70$ then because $N \le 86$, (1) implies that

$$\frac{N-2}{3} + \frac{N-16}{35} \le \frac{N}{5} + \frac{N}{7}$$

which simplifies to $N \le 59$, contradicting $N \ge 70$. it follows that N must be at most 69.

4 POINTS for finding that $N \le 69$.

Checking (1) for $N \le 69$ we find that

when	$N = 69$,	(1) is	$23 + 1 = 13 + 9$,	false
when	$N = 68, 67, 66$,	(1) is	$22 + 1 = 13 + 9$,	false
when	$N = 65$,	(1) is	$21 + 1 = 13 + 9$,	**true**.

Thus the answer is $N = 65$.

1 POINT for concluding.

Second solution.

This is equivalent to find the largest positive integer solution of the equation

$$\left\lfloor \frac{N}{3} \right\rfloor = \left\lfloor \frac{N}{5} \right\rfloor + \left\lfloor \frac{N}{7} \right\rfloor - \left\lfloor \frac{N}{35} \right\rfloor, \tag{1}$$

1 POINT for equality (1).

Let $N = 35k + r$ $(0 \le r < 35)$ be a solution of (1). Then (1) can be writen as

$$\left\lfloor \frac{35k + r}{3} \right\rfloor = 11k + \left\lfloor \frac{r}{5} \right\rfloor + \left\lfloor \frac{r}{7} \right\rfloor.$$

1 POINT.

Now $\frac{35k+r-2}{3} \leq \left\lfloor \frac{35k+r}{3} \right\rfloor$, $\lfloor \frac{r}{5} \rfloor \leq \frac{r}{5}$ and $\lfloor \frac{r}{7} \rfloor \leq \frac{r}{7}$. Therefore

$$\frac{35k+r-2}{3} \leq 11k + \frac{r}{5} + \frac{r}{7} = \frac{12r}{35},$$

which implies that $70k \leq r + 70 < 35 + 70 = 105$. Then $k \leq 1$ or equivalently $N \leq 69$.

4 POINTS for finding that $N \leq 69$.

As in the first solution, checking (1) for $N \leq 69$ we find the answer $N = 65$.

1 POINT for concluding.

Remark.
2 POINTS can be given for finding a good upper bound for example $N \leq 100$ (in the first solution we found $n \leq 86$). 6 POINTS for proving that $N \leq 69$.

1 POINT can be given for the correct answer.

Problem 3.

First Solution.

It is easy to see that the intersection $S \cap T$ has $2n$ sides only if the sides of $S \cap T$ alternate: blue, red, blue, red, etc.

1 POINT.

Denote the vertices of $S \cap T$ clockwise as $C_1D_1C_2D_2 \ldots C_nD_n$ so that the sides C_1D_1, $C_2D_2, \ldots C_nD_n$ are blue. Denote the vertices of S by A_1, $A_2, \ldots,$ A_n and the vertices of T by B_1, $B_2, \ldots,$ B_n so that $C_1D_1 \subset B_1B_2, \ldots, C_nD_n \subset B_nB_1$ and $D_nC_1 \subset A_1A_2, \ldots, D_{n-1}C_n \subset A_nA_1$.

It is easy to check that all the triangles $D_nB_1C_1$, $D_1B_2C_2, \ldots, D_{n-1}B_nC_n$ and $C_1A_2D_1$, $C_2A_3D_2, \ldots, C_nA_1D_n$ are similar.

1 POINT.

Therefore,

$$\frac{D_nC_1}{D_nB_1 + B_1C_1} = \frac{D_1C_2}{D_1B_2 + B_2C_2} = \ldots = \frac{D_{n-1}C_n}{D_{n-1}B_n + B_nC_n} =$$
$$= \frac{C_1D_1}{C_1A_2 + A_2D_1} = \frac{C_1D_2}{C_2A_3 + A_3D_2} = \ldots = \frac{C_nD_n}{C_nA_1 + A_1D_n}.$$

1 POINT.

Hence,

$$\frac{D_nC_1 + D_1C_2 + \ldots + D_{n-1}C_n}{D_nB_1 + B_1C_1 + D_1B_2 + B_2C_2 + \ldots + D_{n-1}B_n + B_nC_n} =$$

$$= \frac{C_1D_1 + C_1D_2 + \ldots + C_nD_n}{C_1A_2 + A_2D_1 + C_2A_3 + A_3D_2 + \ldots + C_nA_1 + A_1D_n}. \tag{1}$$

Let $x = D_nC_1 + D_1C_2 + \ldots + D_{n-1}C_n$ and $y = C_1D_1 + C_1D_2 + \ldots + C_nD_n$ then x is the sum of the blue sides of $S \cap T$ and y is the sum of the red sides. If a is the length of a side of S (or T), then the equality (1) can be written in the following form

$$\frac{x}{na - y} = \frac{y}{na - x}.$$

It follows that

$$\begin{aligned} nax - x^2 &= nay - y^2 \\ na(x - y) &= (x + y)(x - y) \\ (na - x - y)(x - y) &= 0. \end{aligned}$$

Since the perimeter $x + y$ of $S \cap T$ is strictly less than the perimeter na of S or T, $na - x - y > 0$. We obtain $x - y = 0$ and $x = y$ q.e.d.

4 POINTS for concluding.

Second Solution.

As in the first solution, $S \cap T$ has $2n$ sides only if the sides of $S \cap T$ alternate.

1 POINT.

Label the vertices of the red n-gon $R_1, R_2, \ldots R_n$ and the vertices of the blue n-gon $B_1, B_2, \ldots, B_n$. Place the n-gons so that the vertices are in the following clockwise order: $B_1, R_1, B_2, R_2, \ldots, B_n, R_n$. Each of these vertices together with the opposite side determines a triangle and all these triangles are similar.

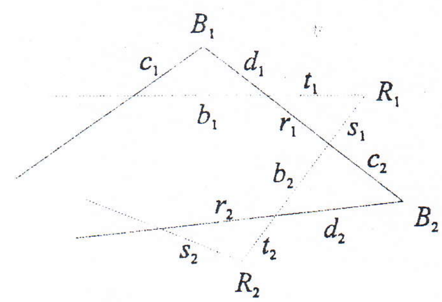
1 POINT.

For each $i = 1, \ldots, n$, we let the lengths of the sides of the triangle determined by B_i be b_i, c_i, d_i in the clockwise order where b_i is the side opposite B_i such that $b_i/b_1 = c_i/c_1 = d_i/d_1 = p_i$. We also let the lengths of the sides of the triangle determined by R_i be r_i, s_i, t_i in the counter clockwise order such that $r_i/b_1 = s_i/c_1 = t_i/d_1 = q_i$.

1 POINT.

Then we want to prove that $b_1 + \cdots + b_n = r_1 + \cdots + r_n$ or $b_1(p_1 + \cdots + p_n) = b_1(q_1 + \cdots + q_n)$, where $p_1 = 1$, or $p = q$ where $p = (p_1 + \cdots + p_n), q = (q_1 + \cdots + q_n)$. The perimeter of the blue n-gon is $c_1 + d_1 + r_1 + \cdots + c_n + d_n + r_n = pc_1 + pd_1 + qb_1$. Likewise the perimeter of the red n-gon is $pb_1 + qc_1 + qd_1$. Equating the two we have $p(c_1 + d_1 - b_1) = q(c_1 + d_1 - b_1)$, which implies $p = q$ as required.

4 POINTS for concluding.



Problem 4.

Answer. All the polynomials of degree 1 with rational coefficients.

First Solution.
Note that a polynomial that satisfies the conditions of the problem takes rational values for rational numbers and irrational values for irrational numbers. Let $f(x)$ be a polynomial of degree n such that $f(r) \in Q$ for every $r \in Q$. For distinct rational numbers $r_0,\ r_1, \ldots, r_n$, where $n = \deg f(x)$ let us define the polynomial

$$\begin{aligned} g(x) &= c_0(x-r_1)(x-r_2)\ldots(x-r_n) + c_1(x-r_0)(x-r_2)\ldots(x-r_n) + \ldots \\ &\quad + c_i(x-r_0)\ldots(x-r_{i-1})(x-r_{i+1})\ldots(x-r_n) + \ldots \\ &\quad + c_n(x-r_0)(x-r_1)\ldots(x-r_{n-1}), \end{aligned} \tag{1}$$

where $c_0,\ c_1, \ldots,\ c_n$ are real numbers.

Suppose that $g(r_i) = f(r_i)$, $i = 0, 1, \ldots, n$. Since $g(r_i) = c_i(r_i - r_0)\ldots(r_i - r_{i-1})(r_i - r_{i+1})\ldots(r_i - r_n)$ then

$$c_i = \frac{g(r_i)}{(r_i - r_0)\ldots(r_i - r_{i-1})(r_i - r_{i+1})} = \frac{f(r_i)}{(r_i - r_0)\ldots(r_i - r_{i-1})(r_i - r_{i+1})}. \tag{2}$$

Clearly c_i is a rational number for $i = 0, 1, \ldots, n$ and therefore the coefficients of $g(x)$ are rational. The polynomial $g(x)$ defined in (1) with coefficients $c_0, c_1, \ldots, c_n$ satisfying (2) coincides with $f(x)$ in $n+1$ points and both polynomials f and g have degree n. It follows that for every real x, $f(x) = g(x)$. Therefore the coefficients of $f(x)$ are rational.

Thus if a polynomial satisfies the conditions, all its coefficients are rational.

1 POINT for proving that the polinomial has rational coefficients.

It is easy to see that all the polynomials of first degree with rational coefficients satisfy the conditions of the problem and polynomials of degree 0 do not satisfy it. Let us prove that no other polynomials exist.

Suppose that $f(x) = a_0 + a_1x + \ldots + a_nx^n$ is a polynomial with rational coefficients and degree $n \geq 2$ that satisfies the conditions of the problem. We may assume that the coefficients of $f(x)$ are integers, because the sets of solutions of equations $f(x) = r$ and $af(x) = ar$, where a is an integer, coincide. Moreover let us denote $g(x) = a_n^{n-1}f(\frac{x}{a_n})$. $g(x)$ is a polynomial with integer coefficients whose leading coefficient is 1. The equation $f(x) = r$ has an irrational root if and only if $g(x) = a_n^{n-1}r$ has an irrational root. Therefore, we may assume WLOG that $f(x)$ has integer coefficients and $a_n = 1$.

1 POINT more for proving that it is suficient to consider polinomials with integer and leading coefficient equal to 1.

Let r be a sufficiently large prime, such that

$$r > \max\{f(1) - f(0), x_1, x_2, \ldots, x_k\},$$

where $\{x_1, x_2, \ldots, x_k\}$ denote the set of all real roots of $f(x) - f(0) - x = 0$. Putting $q = r + f(0) \in Z$, and considering the equality

$$f(x) - q = f(x) - f(0) - r$$

we then have

$$f(1) - q = f(1) - f(0) - r < 0.$$

On the other hand, by the choice of r, we have

$$f(1) - q = f(r) - f(0) - r > 0.$$

It follows from the intermediate value theorem that there is at least one real root p of $f(x) - q = 0$ between 1 and r. Notice that from the criterion theorem for rational roots, the possible positive rational roots of the equation $f(x) - q = f(x) - f(0) - r = 0$ are 1 and r. Thus p must be irrational.

5 POINTS for concluding.

Second Solution.
As in the first solution we may assume WLOG that $f(x)$ has integer coefficients and the leading coefficient is $a_n = 1$.

2 POINTS.

Observing the graph of $f(x)$, it is easy to see that there exists a sufficiently great integer r such that $f(x) = r$ has one possitive root x_0 and for $x \geq x_0$ the derivative $f'(x)$ is greater than 1. The equation $f(x) = r + 1$ has also one positive root $x_1 > x_0$. Since $a_n = 1$, rational roots x_0 and x_1 must be integers. Then $x_1 - x_0 \geq 1$ and

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq 1.$$

It follows that $f'(z) < 1$ for $z \in [x_0, x_1]$. This contradiction proves that $f(x) = r$ necessarily has an irrational root for at least one integer r.

5 POINTS for concluding.

Remark.
No points can be given just for the answer.

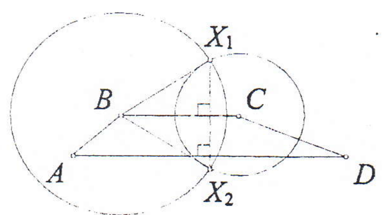
Problem 5.

First Solution.

One of the sides AX_i or BX_i is equal to CD, thus X_i is on one of the circles of radius CD and center A or B. In the same way X_i is on one of circles of radius AB with center C or D. The intersection of these four circles has no more than 8 points so that $n \leq 8$.

1 POINT for finding that $n \leq 8$.

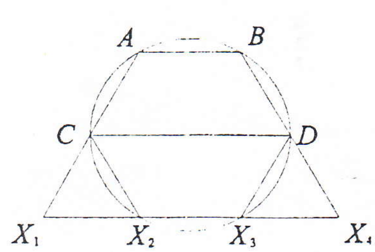
Suppose that circle S_B with center B and radius CD intersects circle S_C with center C and radius AB in two points X_1 and X_2 which satisfy the conditions of the problem. Then in triangles ABX_1 and CDX_1 we have $BX_1 = CD$ and $CX_1 = AB$. Since these triangles are congruent then $AX_1 = DX_1$, therefore X_1 and X_2 are on the perpendicular bisector of AD. On the other hand X_1X_2 is perpendicular to segment BC. Then $BC \| AD$ and AB and CD are the diagonals or nonparallel sides of a trapezoid.



Suppose that $AB < CD$. Then $BX_1 = CD > AB = CX_1$. It follows that the distance from A to the perpendicular bisector of BC must be less than the distance from D to this line otherwise we obtain a contradiction to the condition $AB < CD$. Then for any point X in the perpendicular bisector of BC we have $AX < DX$ and it is not possible to have $AX = CD$, $DX = AB$. Thus if the circle with center A and radius CD intersects the circle with center D and radius AB, then the points of intersection do not satisfy the condition of congruence. Therefore if the points of intersection of S_B with S_C satisfy the condition of congruence, then the points of intersection of S_A with S_D do not. Thus no more than half of the 8 points of intersection of these circles can satisfy the condition of congruence, i.e. $n \leq 4$.

4 POINTS for proving that $n \leq 4$.

If $n = 4$ we have the following example of a regular hexagon.

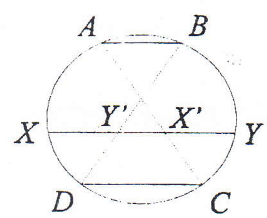


2 POINTS for proving that $n \geq 4$.

Second Solution.

The greatest possible value of n is 4. First we will prove that $n \geq 4$ and finally we will prove that $n \leq 4$.

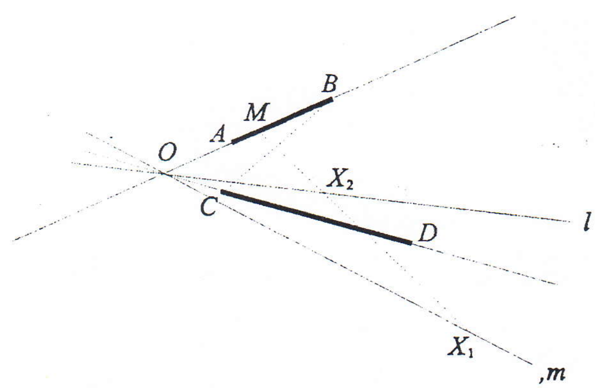
- To prove that $n \geq 4$ it is enough to show a configuration of 8 points that satisfies the conditions of the problem. Let us choose 6 points A, B, Y, C, D, X on a circle such that triangle AYD is equilateral and B, C, X are points on arcs AY, YD, DA respectively, such that arcs AB, YC, DX are equal and less than $60°$. Then it is easy to see that AB, CD and XY are parallel and $\angle DXY = \angle CYX = 60°$.



If XY intersects AC and BD at X', Y', respectively, then $X, Y, X'Y'$ satisfy the conditions of the problem.

2 POINTS for proving that $n \geq 4$.

- To show that $n \leq 4$ let us consider the following figure.



Suppose that $AB < CD$. The trace of point K such that $(ABK) = (CDK)$ is two lines l and m through the point of intersection O of AB with CD. (If $AB \| CD$ then the trace of K is formed by two parallel lines.)

If X is a point such that $\triangle ABX \cong \triangle CDX$ then X must be on one or more of the perpendicular bisectors of the segments AC, AD, BC, BD. Since X lies on l or m, on each of the perpendicular bisectors of AC, AD, BC, BD there can be no more than 2 intersection points, i.e. $n \leq 8$.

1 POINT for proving that $n \leq 8$.

We will prove that at most one point on each perpendicular bisector satisfies the conditions of the problem.

WLOG Suposse that X_1 and X_2 are points on the perpendicular bisector of BC such that $\triangle ABX_1 \cong \triangle CDX_1$ and $\triangle ABX_2 \cong \triangle CDX_2$ with $X_1 \in l$ and $X_2 \in m$.

Let us prove that $AX_1 = CD$. Since $AB \neq CD$ one of the segments AX_1 or BX_1 is equal to CD. If $BX_1 = CD$ then $CX_1 = CD$ by construction, and $\triangle CDX_1$ has two sides equal to CD, thus $AX_1 = CD$.

In the same way we have $DX_1 = AB$

The same argument can be used if we consider the point X_2. In this case we conclude that $AX_1 = AX_2$ and $DX_1 = DX_2$, then AD and X_1X_2 are perpendicular. But BC and X_1X_2 perpendicular and therefore $AD \| BC$. We are considering the case when AD and BC are diagonals of a cuadrilateral, therefore they are not parallel. This is a contradiction.

If we consider the perpendicular bisector of BD, the same argument allows us to conclude that AC and BD are parallel. Therefore a point X on l or m (i.e $(XAB) = (XCD)$) satisfies $(XOB) = (XOD)$ because triangles OAC and OBD are similar. Thus, the points of intersection of AC with l and BD with l are midpoints of the respective segments. Then $X_1X_2 \subset l$ and, since l is the perpendicular bisector and median, triangle ABD is isosceles with $OB = OC$. So, $AB = CD$, which is a contradiction. Therefore $n \leq 4$.

4 POINT for proving that $n \leq 4$.