APMO 2022 Solution

1. Find all pairs (a, b) of positive integers such that a^3 is a multiple of b^2 and $b-1$ is a multiple of $a-1$.

   Note: An integer n is said to be a multiple of an integer m if there is an integer k such that $n = km$.

   Solution

   Solution 1.1

   By inspection, we see that the pairs (a, b) with $a = b$ are solutions, and so too are the pairs $(a, 1)$. We will see that these are the only solutions.

   - *Case 1.* Consider the case $b < a$. Since $b - 1$ is a multiple of $a - 1$, it follows that $b = 1$. This yields the second set of solutions described above.
   - *Case 2.* This leaves the case $b \geq a$. Since the positive integer a^3 is a multiple of b^2, there is a positive integer c such that $a^3 = b^2 c$.

     Note that $a \equiv b \equiv 1$ modulo $a - 1$. So we have

     $$1 \equiv a^3 = b^2 c \equiv c \pmod{a-1}.$$

     If $c < a$, then we must have $c = 1$, hence, $a^3 = b^2$. So there is a positive integer d such that $a = d^2$ and $b = d^3$. Now $a - 1 \mid b - 1$ yields $d^2 - 1 \mid d^3 - 1$. This implies that $d + 1 \mid d(d+1) + 1$, which is impossible.

     If $c \geq a$, then $b^2 c \geq b^2 a \geq a^3 = b^2 c$. So there's equality throughout, implying $a = c = b$. This yields the first set of solutions described above.

   Therefore, the solutions described above are the only solutions.

   □

   Solution 1.2

   We will start by showing that there are positive integers x, c, d such that $a = x^2 cd$ and $b = x^3 c$. Let $g = \gcd(a, b)$ so that $a = gd$ and $b = gx$ for some coprime d and x. Then, $b^2 \mid a^3$ is equivalent to $g^2 x^2 \mid g^3 d^3$, which is equivalent to $x^2 \mid gd^3$. Since x and d are coprime, this implies $x^2 \mid g$. Hence, $g = x^2 c$ for some c, giving $a = x^2 cd$ and $b = x^3 c$ as required.

   Now, it remains to find all positive integers x, c, d satisfying

   $$x^2 cd - 1 \mid x^3 c - 1.$$

   That is, $x^3 c \equiv 1 \pmod{x^2 cd - 1}$. Assuming that this congruence holds, it follows that $d \equiv x^3 cd \equiv x \pmod{x^2 cd - 1}$. Then, either $x = d$ or $x - d \geq x^2 cd - 1$ or $d - x \geq x^2 cd - 1$.

   - If $x = d$ then $b = a$.
   - If $x - d \geq x^2 cd - 1$, then $x - d \geq x^2 cd - 1 \geq x - 1 \geq x - d$. Hence, each of these inequalities must in fact be an equality. This implies that $x = c = d = 1$, which implies that $a = b = 1$.
   - If $d - x \geq x^2 cd - 1$, then $d - x \geq x^2 cd - 1 \geq d - 1 \geq d - x$. Hence, each of these inequalities must in fact be an equality. This implies that $x = c = 1$, which implies that $b = 1$.

   Hence the only solutions are the pairs (a, b) such that $a = b$ or $b = 1$. These pairs can be checked to satisfy the given conditions.

$\square$

Solution 1.3

All answers are (n, n) and $(n, 1)$ where n is any positive integer. They all clearly work.

To show that these are all solutions, note that we can easily eliminate the case $a = 1$ or $b = 1$. Thus, assume that $a, b \neq 1$ and $a \neq b$. By the second divisibility, we see that $a - 1 \mid b - a$. However, $\gcd(a, b) \mid b - a$ and $a - 1$ is relatively prime to $\gcd(a, b)$. This implies that $(a - 1)\gcd(a, b) \mid b - a$, which implies $\gcd(a, b) \,\Big|\, \dfrac{b-1}{a-1} - 1$.

The last relation implies that $\gcd(a, b) < \dfrac{b-1}{a-1}$, since the right-hand side are positive. However, due to the first divisibility,

$$\gcd(a, b)^3 = \gcd(a^3, b^3) \geq \gcd(b^2, b^3) = b^2.$$

Combining these two inequalities, we get that

$$b^{\frac{2}{3}} < \frac{b-1}{a-1} < 2\frac{b}{a}.$$

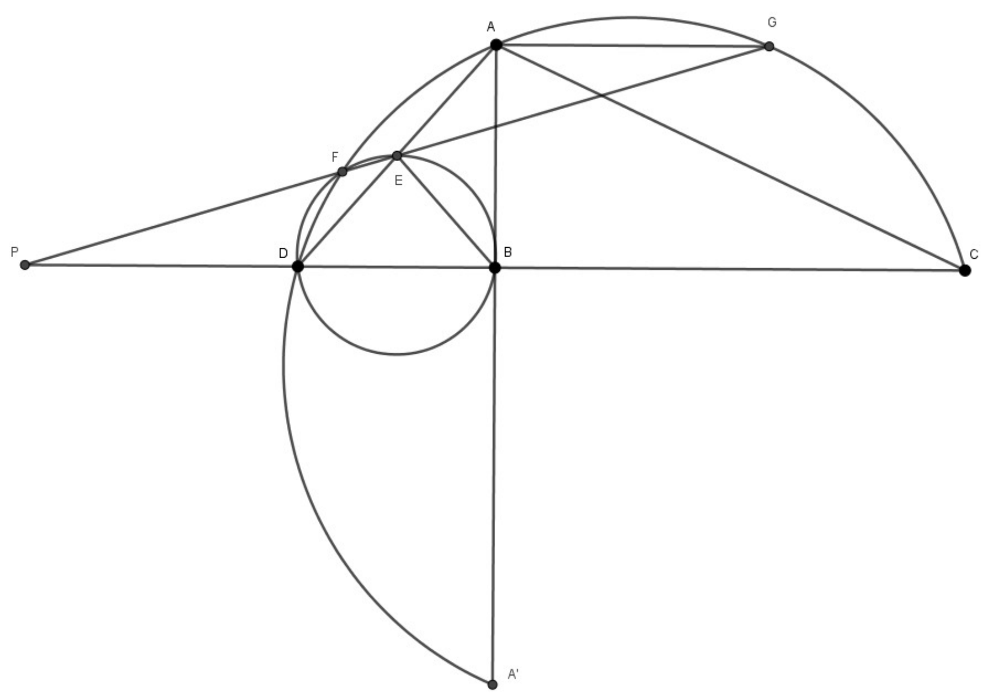
This implies $a < 2b^{\frac{1}{3}}$. However, $b^2 \mid a^3$ gives $b \leq a^{\frac{3}{2}}$. This forces

$$a < 2(a^{\frac{3}{2}})^{\frac{1}{3}} = 2\sqrt{a} \implies a < 4.$$

Extracting $a = 2, 3$ by hand yields no additional solution.

$\square$

2. Let ABC be a right triangle with $\angle B = 90^\circ$. Point D lies on the line CB such that B is between D and C. Let E be the midpoint of AD and let F be the second intersection point of the circumcircle of $\triangle ACD$ and the circumcircle of $\triangle BDE$. Prove that as D varies, the line EF passes through a fixed point.



Solution

Solution 2.1

Let the line EF intersect the line BC at P and the circumcircle of $\triangle ACD$ at G distinct from F. We will prove that P is the fixed point.

First, notice that $\triangle BED$ is isosceles with $EB = ED$. This implies $\angle EBC = \angle EDP$.

Then, $\angle DAG = \angle DFG = \angle EBC = \angle EDP$ which implies $AG \parallel DC$. Hence, $AGCD$ is an isosceles trapezoid.

Also, $AG \parallel DC$ and $AE = ED$. This implies $\triangle AEG \cong \triangle DEP$ and $AG = DP$.

Since B is the foot of the perpendicular from A onto the side CD of the isosceles trapezoid $AGCD$, we have $PB = PD + DB = AG + DB = BC$, which does not depend on the choice of D. Hence, the initial statement is proven.

Solution 2.2

Set up a coordinate system where BC is along the positive x-axis, BA is along the positive y-axis, and B is the origin. Take $A = (0, a)$, $B = (0, 0)$, $C = (c, 0)$, $D = (-d, 0)$ where $a, b, c, d > 0$. Then $E = (-\frac{d}{2}, \frac{a}{2})$. The general equation of a circle is

$$x^2 + y^2 + 2fx + 2gy + h = 0 \tag{1}$$

Substituting the coordinates of A, D, C into (1) and solving for f, g, h, we find that the equation of the circumcircle of $\triangle ADC$ is

$$x^2 + y^2 + (d - c)x + (\frac{cd}{a} - a)y - cd = 0. \tag{2}$$

Similarly, the equation of the circumcircle of $\triangle BDE$ is

$$x^2 + y^2 + dx + (\frac{d^2}{2a} - \frac{a}{2})y = 0. \tag{3}$$

Then (3)−(2) gives the equation of the line DF which is

$$cx + \frac{a^2 + d^2 - 2cd}{2a}y + cd = 0. \tag{4}$$

Solving (3) and (4) simultaneously, we get

$$F = \left(\frac{c(d^2 - a^2 - 2cd)}{a^2 + (d - 2c)^2}, \frac{2ac(c - d)}{a^2 + (d - 2c)^2} \right),$$

and the other solution $D = (-d, 0)$.

From this we obtain the equation of the line EF which is $ax + (d - 2c)y + ac = 0$. It passes through $P(-c, 0)$ which is independent of d.

3. Find all positive integers $k < 202$ for which there exists a positive integer n such that

$$\left\{\frac{n}{202}\right\} + \left\{\frac{2n}{202}\right\} + \cdots + \left\{\frac{kn}{202}\right\} = \frac{k}{2},$$

where $\{x\}$ denote the fractional part of x.

Note: $\{x\}$ denotes the real number k with $0 \leq k < 1$ such that $x - k$ is an integer.

Solution

Denote the equation in the problem statement as (*), and note that it is equivalent to the condition that the average of the remainders when dividing n, $2n$, …, kn by 202 is 101. Since $\left\{\frac{in}{202}\right\}$ is invariant in each residue class modulo 202 for each $1 \leq i \leq k$, it suffices to consider $0 \leq n < 202$.

If $n = 0$, so is $\left\{\frac{in}{202}\right\}$, meaning that (*) does not hold for any k. If $n = 101$, then it can be checked that (*) is satisfied if and only if $k = 1$. From now on, we will assume that $101 \nmid n$.

For each $1 \le i \le k$, let $a_i = \left\lfloor \frac{in}{202} \right\rfloor = \frac{in}{202} - \left\{\frac{in}{202}\right\}$. Rewriting (*) and multiplying the equation by 202, we find that

$$n(1 + 2 + \ldots + k) - 202(a_1 + a_2 + \ldots + a_k) = 101k.$$

Equivalently, letting $z = a_1 + a_2 + \ldots + a_k$,

$$nk(k + 1) - 404z = 202k.$$

Since n is not divisible by 101, which is prime, it follows that $101 \mid k(k + 1)$. In particular, $101 \mid k$ or $101 \mid k + 1$. This means that $k \in \{100, 101, 201\}$. We claim that all these values of k work.

- If $k = 201$, we may choose $n = 1$. The remainders when dividing n, $2n$, ..., kn by 202 are 1, 2, ..., 201, which have an average of 101.
- If $k = 100$, we may choose $n = 2$. The remainders when dividing n, $2n$, ..., kn by 202 are 2, 4, ..., 200, which have an average of 101.
- If $k = 101$, we may choose $n = 51$. To see this, note that the first four remainders are 51, 102, 153, 2, which have an average of 77. The next four remainders (53, 104, 155, 4) are shifted upwards from the first four remainders by 2 each, and so on, until the 25th set of the remainders (99, 150, 201, 50) which have an average of 125. Hence, the first 100 remainders have an average of $\frac{77 + 125}{2} = 101$. The 101th remainder is also 101, meaning that the average of all 101 remainders is 101.

In conclusion, all values $k \in \{1, 100, 101, 201\}$ satisfy the initial condition.

□

4. Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labelled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i + 1$. Cathy wins if at any point there is a box containing only marble n.
   Determine all pairs of integers (n, k) such that Cathy can win this game.

   Solution

   We claim Cathy can win if and only if $n \le 2^{k-1}$.

   First, note that each non-empty box always contains a consecutive sequence of labeled marbles. This is true since Cathy is always either removing from or placing in the lowest marble in a box. As a consequence, every move made is reversible.

   Next, we prove by induction that Cathy can win if $n = 2^{k-1}$. The base case of $n = k = 1$ is trivial. Assume a victory can be obtained for m boxes and 2^{m-1} marbles. Consider the case of $m + 1$ boxes and 2^m marbles. Cathy can first perform a sequence of moves so that only marbles $2^{m-1}, \ldots, 2^m$ are left in the starting box, while keeping one box, say B, empty. Now move the marble 2^{m-1} to box B, then reverse all of the initial moves while treating B as the starting box. At the end of that, we will have marbles $2^{m-1} + 1$, ..., 2^m in the starting box, marbles 1, 2, ..., 2^{m-1} in box B, and $m - 1$ empty boxes. By repeating the original sequence of moves on marbles $2^{m-1} + 1, \ldots, 2^m$, using the m boxes that are not box B, we can reach a state where only marble 2^m remains in the starting box. Therefore a victory is possible if $n = 2^{k-1}$ or smaller.

We now prove by induction that Cathy loses if $n = 2^{k-1} + 1$. The base case of $n = 2$ and $k = 1$ is trivial. Assume a victory is impossible for m boxes and $2^{m-1}+1$ marbles. For the sake of contradiction, suppose that victory is possible for $m+1$ boxes and $2^m + 1$ marbles. In a winning sequence of moves, consider the last time a marble $2^{m-1} + 1$ leaves the starting box, call this move X. After X, there cannot be a time when marbles $1, \dots, 2^{m-1} + 1$ are all in the same box. Otherwise, by reversing these moves after X and deleting marbles greater than $2^{m-1} + 1$, it gives us a winning sequence of moves for $2^{m-1} + 1$ marbles and m boxes (as the original starting box is not used here), contradicting the inductive hypothesis. Hence starting from X, marbles 1 will never be in the same box as any marbles greater than or equal to $2^{m-1} + 1$.

Now delete marbles $2, \dots, 2^{m-1}$ and consider the winning moves starting from X. Marble 1 would only move from one empty box to another, while blocking other marbles from entering its box. Thus we effectively have a sequence of moves for $2^{m-1} + 1$ marbles, while only able to use m boxes. This again contradicts the inductive hypothesis. Therefore, a victory is not possible if $n = 2^{k-1} + 1$ or greater.

$\square$

5. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Determine the minimum value of $(a - b)(b - c)(c - d)(d - a)$ and determine all values of (a, b, c, d) such that the minimum value is achieved.

   Solution

   The minimum value is $\boxed{-\frac{1}{8}}$. There are eight equality cases in total. The first one is

$$\left(\frac{1}{4} + \frac{\sqrt{3}}{4}, -\frac{1}{4} - \frac{\sqrt{3}}{4}, \frac{1}{4} - \frac{\sqrt{3}}{4}, -\frac{1}{4} + \frac{\sqrt{3}}{4}\right).$$

   Cyclic shifting all the entries give three more quadruples. Moreover, flipping the sign ($(a, b, c, d) \to (-a, -b, -c, -d)$) all four entries in each of the four quadruples give four more equality cases.

   Solution 5.1

   Since the expression is cyclic, we could WLOG $a = \max\{a, b, c, d\}$. Let

$$S(a, b, c, d) = (a - b)(b - c)(c - d)(d - a)$$

   Note that we have given (a, b, c, d) such that $S(a, b, c, d) = -\frac{1}{8}$. Therefore, to prove that $S(a, b, c, d) \geq -\frac{1}{8}$, we just need to consider the case where $S(a, b, c, d) < 0$.

   - **Exactly 1 of $a - b, b - c, c - d, d - a$ is negative.**
     Since $a = \max\{a, b, c, d\}$, then we must have $d - a < 0$. This forces $a > b > c > d$. Now, let us write
     $$S(a, b, c, d) = -(a - b)(b - c)(c - d)(a - d)$$
     Write $a - b = y, b - c = x, c - d = w$ for some positive reals $w, x, y > 0$. Plugging to the original condition, we have
     $$(d + w + x + y)^2 + (d + w + x)^2 + (d + w)^2 + d^2 - 1 = 0 \ (*)$$
     and we want to prove that $wxy(w + x + y) \leq \frac{1}{8}$. Consider the expression $(*)$ as a quadratic in d:
     $$4d^2 + d(6w + 4x + 2y) + ((w + x + y)^2 + (w + x)^2 + w^2 - 1) = 0$$

Since d is a real number, then the discriminant of the given equation has to be non-negative, i.e. we must have

$$\begin{aligned} 4 &\geq 4((w+x+y)^2 + (w+x)^2 + w^2) - (3w+2x+y)^2 \\ &= (3w^2 + 2wy + 3y^2) + 4x(w+x+y) \\ &\geq 8wy + 4x(w+x+y) \\ &= 4(x(w+x+y) + 2wy) \end{aligned}$$

However, AM-GM gives us

$$wxy(w+x+y) \leq \frac{1}{2}\left(\frac{x(w+x+y)+2wy}{2}\right)^2 \leq \frac{1}{8}$$

This proves $S(a,b,c,d) \geq -\frac{1}{8}$ for any $a,b,c,d \in \mathbb{R}$ such that $a > b > c > d$. Equality holds if and only if $w = y$, $x(w+x+y) = 2wy$ and $wxy(w+x+y) = \frac{1}{8}$. Solving these equations gives us $w^4 = \frac{1}{16}$ which forces $w = \frac{1}{2}$ since $w > 0$. Solving for x gives us $x(x+1) = \frac{1}{2}$, and we will get $x = -\frac{1}{2} + \frac{\sqrt{3}}{2}$ as $x > 0$. Plugging back gives us $d = -\frac{1}{4} - \frac{\sqrt{3}}{4}$, and this gives us

$$(a,b,c,d) = \left(\frac{1}{4} + \frac{\sqrt{3}}{4}, -\frac{1}{4} + \frac{\sqrt{3}}{4}, \frac{1}{4} - \frac{\sqrt{3}}{4}, -\frac{1}{4} - \frac{\sqrt{3}}{4}\right)$$

Thus, any cyclic permutation of the above solution will achieve the minimum equality.

- **Exactly 3 of $a-b, b-c, c-d, d-a$ are negative** Since $a = \max\{a,b,c,d\}$, then $a-b$ has to be positive. So we must have $b < c < d < a$. Now, note that

$$\begin{aligned} S(a,b,c,d) &= (a-b)(b-c)(c-d)(d-a) \\ &= (a-d)(d-c)(c-b)(b-a) \\ &= S(a,d,c,b) \end{aligned}$$

Now, note that $a > d > c > b$. By the previous case, $S(a,d,c,b) \geq -\frac{1}{8}$, which implies that

$$S(a,b,c,d) = S(a,d,c,b) \geq -\frac{1}{8}$$

as well. Equality holds if and only if

$$(a,b,c,d) = \left(\frac{1}{4} + \frac{\sqrt{3}}{4}, -\frac{1}{4} - \frac{\sqrt{3}}{4}, \frac{1}{4} - \frac{\sqrt{3}}{4}, -\frac{1}{4} + \frac{\sqrt{3}}{4}\right)$$

and its cyclic permutation.

Solution 5.2

The minimum value is $\boxed{-\frac{1}{8}}$. There are eight equality cases in total. The first one is

$$\left(\frac{1}{4} + \frac{\sqrt{3}}{4}, -\frac{1}{4} - \frac{\sqrt{3}}{4}, \frac{1}{4} - \frac{\sqrt{3}}{4}, -\frac{1}{4} + \frac{\sqrt{3}}{4}\right).$$

Cyclic shifting all the entries give three more quadruples. Moreover, flipping the sign ($(a,b,c,d) \to (-a,-b,-c,-d)$) all four entries in each of the four quadruples give four more equality cases. We then begin the proof by the following optimization:

Claim 1. In order to get the minimum value, we must have $a+b+c+d = 0$.

Proof. Assume not, let $\delta = \frac{a+b+c+d}{4}$ and note that

$$(a-\delta)^2 + (b-\delta)^2 + (c-\delta)^2 + (d-\delta)^2 < a^2 + b^2 + c^2 + d^2,$$

so by shifting by δ and scaling, we get an even smaller value of $(a-b)(b-c)(c-d)(d-a)$.

□

The key idea is to substitute the variables

$$\begin{aligned} x &= ac + bd \\ y &= ab + cd \\ z &= ad + bc, \end{aligned}$$

so that the original expression is just $(x-y)(x-z)$. We also have the conditions $x, y, z \geq -0.5$ because of:

$$2x + (a^2 + b^2 + c^2 + d^2) = (a+c)^2 + (b+d)^2 \geq 0.$$

Moreover, notice that

$$0 = (a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2(x+y+z) \implies x + y + z = \frac{-1}{2}.$$

Now, we reduce to the following optimization problem.

Claim 2. Let $x, y, z \geq -0.5$ such that $x + y + z = -0.5$. Then, the minimum value of

$$(x-y)(x-z)$$

is $-1/8$. Moreover, the equality case occurs when $x = -1/4$ and $\{y, z\} = \{1/4, -1/2\}$.

Proof. We notice that

$$\begin{aligned} (x-y)(x-z) + \frac{1}{8} &= \left(2y + z + \frac{1}{2}\right)\left(2z + y + \frac{1}{2}\right) + \frac{1}{8} \\ &= \frac{1}{8}(4y + 4z + 1)^2 + \left(y + \frac{1}{2}\right)\left(z + \frac{1}{2}\right) \geq 0. \end{aligned}$$

The last inequality is true since both $y + \frac{1}{2}$ and $z + \frac{1}{2}$ are not less than zero.

The equality in the last inequality is attained when either $y + \frac{1}{2} = 0$ or $z + \frac{1}{2} = 0$, and $4y + 4z + 1 = 0$. This system of equations give $(y, z) = (1/4, -1/2)$ or $(y, z) = (-1/2, 1/4)$ as the desired equality cases.

□

Note: We can also prove (the weakened) Claim 2 by using Lagrange Multiplier, as follows. We first prove that, in fact, $x, y, z \in [-0.5, 0.5]$. This can be proved by considering that

$$-2x + (a^2 + b^2 + c^2 + d^2) = (a-c)^2 + (b-d)^2 \geq 0.$$

We will prove the Claim 2, only that in this case, $x, y, z \in [-0.5, 0.5]$. This is already sufficient to prove the original question. We already have the bounded domain $[-0.5, 0.5]^3$, so the global minimum must occur somewhere. Thus, it suffices to consider two cases:

- If the global minimum lies on the boundary of $[-0.5, 0.5]^3$. Then, one of x, y, z must be -0.5 or 0.5. By symmetry between y and z, we split to a few more cases.

- If $x = 0.5$, then $y = z = -0.5$, so $(x-y)(x-z) = 1$, not the minimum.
- If $x = -0.5$, then both y and z must be greater or equal to x, so $(x-y)(x-z) \geq 0$, not the minimum.
- If $y = 0.5$, then $x = z = -0.5$, so $(x-y)(x-z) = 0$, not the minimum.
- If $y = -0.5$, then $z = -x$, so

$$(x-y)(x-z) = 2x(x+0.5),$$

which obtain the minimum at $x = -1/4$. This gives the desired equality case.

- If the global minimum lies in the interior $(-0.5, 0.5)^3$, then we apply Lagrange multiplier:

$$\begin{aligned}
\frac{\partial}{\partial x}(x-y)(x-z) = \lambda \frac{\partial}{\partial x}(x+y+z) &\implies 2x - y - z = \lambda. \\
\frac{\partial}{\partial y}(x-y)(x-z) = \lambda \frac{\partial}{\partial y}(x+y+z) &\implies z - x = \lambda. \\
\frac{\partial}{\partial z}(x-y)(x-z) = \lambda \frac{\partial}{\partial z}(x+y+z) &\implies y - x = \lambda.
\end{aligned}$$

Adding the last two equations gives $\lambda = 0$, or $x = y = z$. This gives $(x-y)(x-z) = 0$, not the minimum.

Having exhausted all cases, we are done.